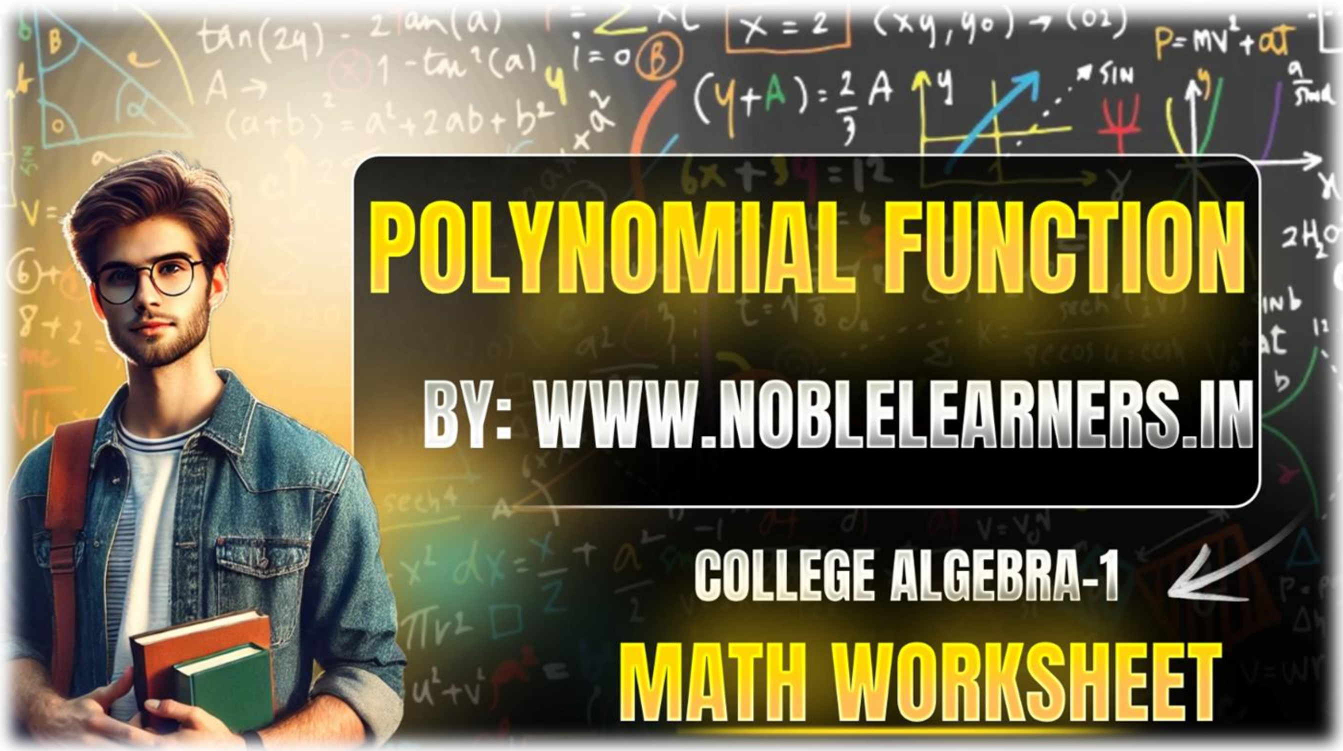

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2.1 Exercises

1. How can the vertex of a parabola be used in solving real world problems?

2. Explain why the condition of $a \neq 0$ is imposed in the definition of the quadratic function.

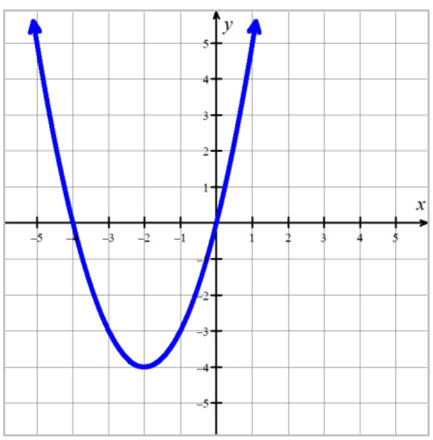
3. Graph the function $g(x) = (x-3)^2 + 4$ by starting with the graph of $f(x) = x^2$ and applying transformations. Identify the vertex of $g(x)$.

4. Graph the function $g(x) = -2(x+4)^2$ by starting with the graph of $f(x) = x^2$ and applying transformations. Identify the vertex of $g(x)$.

In Exercises 5 and 6, the graph of a quadratic function is given. Identify the equation, in standard form, of the function that has been graphed.

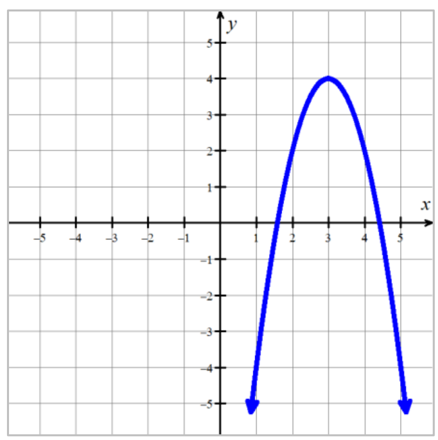
5.

Figure Ex2.1. 1



6.

Figure Ex2.1. 2



In Exercises 7 – 10, convert the function from general form to standard form.

7. $f(x) = x^2 - 2x - 8$

8. $f(x) = x^2 - 6x - 1$

9. $f(x) = 2x^2 - 4x + 7$

10. $f(x) = -4x^2 - 12x + 3$

In Exercises 11 – 26, find the vertex, the axis of symmetry and any *x*- or *y*-intercepts. Graph each function and determine its range.

11. $f(x) = x^2 + 2$

12. $f(x) = -(x+2)^2$

13. $f(x) = -(x+2)^2 + 2$

14. $f(x) = x^2 - 2x$

15. $f(x) = x^2 - 2x - 8$

16. $f(x) = x^2 - 6x - 1$

17. $f(x)=x^2-5x-6$

18. $f(x)=x^2-7x+3$

19. $f(x)=-2(x+1)^2+4$

20. $f(x)=2x^2-4x-1$

21. $f(x)=2x^2-4x+7$

22. $f(x)=-2x^2+5x-8$

23. $f(x)=4x^2-12x-3$

24. $f(x)=-3x^2+4x-7$

25. $f(x)=x^2+x+1$

26. $f(x)=-3x^2+5x+4$

In Exercises 27 – 32, find the real solution(s) of the given equation.

27. $(x+3)^2-4(x+3)-5=0$

28. $\left(\frac{1}{x+1}\right)^2-2\left(\frac{1}{x+1}\right)-3=0$

29. $x-3\sqrt{x}+2=0$

30. $4x^4+9=13x^2$

31. $2x^{-2}=x^{-1}+1$

32. $y^{\frac{1}{3}}+y^{\frac{1}{6}}-2=0$

In Exercises 33 – 37, the profit function $P(x)$ is given.

- Find the number of items that need to be sold in order to maximize profit. Be sure that your answer is reasonable in the context of the problem.
- Find the maximum profit.
- Find and interpret the zeros of $P(x)$.

33. The profit, in dollars, made by selling x "I'd rather be a Sasquatch" t-shirts is $P(x)=-2x^2+28x-26$, $0\le x\le 15$.

34. The profit, in dollars, made by selling x bottles of 100% all-natural certified free-trade organic Sasquatch Tonic is $P(x)=-x^2+25x-100$, $0\le x\le 35$.

35. The profit, in cents, made by selling x cups of Mountain Thunder Lemonade at Junior's lemonade stand is $P(x)=-3x^2+72x-240$, $0\le x\le 30$.

36. The daily profit, in dollars, made by selling x Sasquatch Berry Pies is $P(x)=-0.5x^2+9x-36$, $0\le x\le 24$.

37. The monthly profit, in hundreds of dollars, made by selling x custom built electric scooters is $P(x)=-2x^2+120x-1000$, $0\le x\le 70$.

38. Using data from the Bureau of Transportation statistics, the average fuel economy F (in miles per gallon) for passenger cars in the US can be modeled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \le t \le 28$, where t is the number of years since 1980. Find and interpret the coordinates of the vertex of the graph of $y = F(t)$.

39. The temperature T (in degrees Fahrenheit), t hours after 6 AM, is given by $T(t) = -\frac{1}{2}t^2 + 8t + 32$, $0 \le t \le 12$. What is the warmest temperature of the day? When does this happen?

40. Suppose $C(x) = x^2 - 10x + 27$, $3 \le x \le 15$, represents the marginal cost (in hundreds of dollars) to produce an additional x thousand pens. How many additional pens should be produced to minimize the marginal cost? What is the minimum marginal cost?

41. Suppose $h(x) = -\frac{1}{200}x^2 + \frac{4}{5}x + 3$ represents a baseball's height above the ground where x is the baseball's horizontal distance from the home plate. Both x and h are measured in feet. What is the maximum height of the ball and at what distance from the home plate does it occur?

42. Dani wishes to plant a vegetable garden along one side of her house. She wants the plot to be rectangular to simplify landscaping maintenance. In her garage, she has 32 linear feet of fencing. Since one side of the garden will border the house, Dani doesn't need fencing along that side. What are the dimensions of the garden that will maximize the area of the garden? What is the maxium area of the garden?

43. In the situation of **Example 2.1.6**, Kyle has a nightmare that one of his alpacas fell into the stream and was injured. To avoid this, he wants to move his rectangular pasture away from the stream. This means that all four sides of the pasture require fencing. If the total amount of fencing available is still 200 linear feet, what dimensions will now maximize the area of the pasture? What is the maximum area? Assuming an average alpaca requires 25 square feet of pasture, how many alpacas can he raise?

44. What is the largest rectangular area one can enclose with 14 inches of string?

45. The height of an object dropped from the roof of an eight story building is modeled by $h(t) = -16t^2 + 64$, $0 \le t \le 2$. Here, h is the height (in feet) of the object above the ground, t seconds after the object is dropped. How long, after being dropped, is it before the object hits the ground?

46. The height h (in feet) of a model rocket above the ground, t seconds after lift-off, is given by $h(t) = -5t^2 + 100t$, $0 \le t \le 20$. When does the rocket reach its maximum height above the ground? What is its maximum height?

47. Jason participates in the Highland Games. In one event, the hammer throw, the height h (in feet) of the hammer above the ground t seconds after Jason lets it go is modeled by $h(t) = -16t^2 + 22.08t + 6$. What is the hammer's maximum height? What is the hammer's total time in the air? Round your answers to two decimal places.

48. Assuming no air resistance or forces other than the Earth's gravity, the height above the ground at time t (in seconds) of a falling object is given by $s(t) = -4.9t^2 + v_0 t + s_0$, where s is in meters, v_0 (the object's initial velocity) is in meters per second, and s_0 (the initial height of the object) is in meters.

(a) What is the applied domain of this function?

(b) Discuss with your classmates what each of $v_0 > 0$, $v_0 = 0$ and $v_0 < 0$ would mean.

(c) Come up with a scenario in which $s_0 < 0$.

(d) Let's say a slingshot is used to shoot a marble straight up from the ground ($s_0 = 0$) with an initial velocity of 15 meters per second. What is the marble's maximum height above the ground? At what time will it hit the ground?

(e) Now shoot the marble from the top of a tower that is 25 meters tall. When does the marble hit the ground?

(f) What would the height function be if, instead of shooting the marble up off the top of the tower, you were to shoot it straight DOWN from the top of the tower?

49. The two towers of a suspension bridge are 400 feet apart. The parabolic cable attached to the tops of the towers is 10 feet above the point on the bridge deck that is midway between the towers. If the towers are 100 feet tall, find the height of the cable directly above a point of the bridge deck that is 50 feet to the right of the left-hand tower.

50. Find all points on the line $y = 1 - x$ that are 2 units away from $(1, -1)$.

51. Let L be the line $y = 2x + 1$. Find a function $D(x)$ that measures the distance squared from a point on L to $(0,0)$. Use this to find the point on L closest to $(0,0)$.

52. With the help of your classmates, show that if a quadratic function $f(x) = ax^2 + bx + c$ has two real zeros, then the *x*-coordinate of the vertex is the midpoint of the zeros.

2.2 Exercises

1. If a polynomial function is in factored form, what first step would be useful to determine the degree of the function?
2. In general, explain the end behavior of the graph of a polynomial function with odd degree if the leading coefficient is positive.

In Exercises 3 – 12, find the degree, leading term, leading coefficient, constant term, and end behavior of the given polynomial.

3. $f(x)=4-x-3x^2$
4. $g(x)=3x^5-2x^2+x+1$
5. $q(r)=1-16r^4$
6. $Z(b)=42b-b^3$
7. $f(x)=\sqrt{3}x^{17}+22.5x^{10}-\pi x^7+\frac{1}{3}$
8. $s(t)=-4.9t^2+v_0t+s_0$
9. $P(x)=(x-1)(x-2)(x-3)(x-4)$
10. $p(t)=-t^2(3-5t)(t^2+t+4)$
11. $f(x)=-2x^3(x+1)(x+2)^2$
12. $G(t)=4(t-2)^2\left(t+\frac{1}{2}\right)$

In Exercises 13 – 22, find the *x*-intercept(s) of the given polynomial, the multiplicities of the corresponding zeros, and the *y*-intercept. Use this information, along with end behavior where necessary, to sketch a rough graph of the polynomial.

13. $a(x)=x(x+2)^2$
14. $g(x)=x(x+2)^3$
15. $f(x)=-2(x-2)^2(x+1)$
16. $g(x)=(2x+1)^2(x-3)$
17. $F(x)=x^3(x+2)^2$
18. $P(x)=(x-1)(x-2)(x-3)(x-4)$
19. $Q(x)=(x+5)^2(x-3)^4$
20. $h(x)=x^2(x-2)^2(x+2)^2$
21. $H(t)=(3-t)(t+1)^2$
22. $Z(b)=b(49-b^2)$

In Exercises 23 – 27, create a polynomial p that has the desired characteristics. You may leave the polynomial in factored form.

23. ▪ The *x*-intercepts of p are $(\pm1,0)$ and $(\pm2,0)$.
    ▪ The leading term of $p(x)$ is $117x^4$.

24. ▪ The zeros of p are $x = 1$ and $x = 3$.

    ▪ $x = 3$ is a zero of multiplicity 2.

    ▪ $p(x)$ has the *y*-intercept $(0, 45)$.

25. ▪ The solutions to $p(x) = 0$ are $x = \pm 3$ and $x = 6$.

    ▪ The leading term of $p(x)$ is $7x^4$.

    ▪ The point $(-3, 0)$ is a local minimum on the graph of $y = p(x)$.

26. ▪ The solutions to $p(x) = 0$ are $x = \pm 3$, $x = -2$ and $x = 4$.

    ▪ The leading term of $p(x)$ is $-x^5$.

    ▪ The point $(-2, 0)$ is a local maximum on the graph of $y = p(x)$.

27. ▪ p is of degree 4.

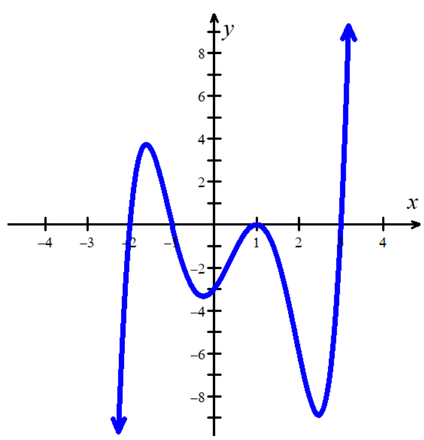
    ▪ As $x \to \infty$, $p(x) \to -\infty$.

    ▪ p has exactly three *x*-intercepts: $(-6, 0)$, $(1, 0)$ and $(117, 0)$.

    ▪ The graph of $y = p(x)$ touches the *x*-axis at $(1, 0)$.

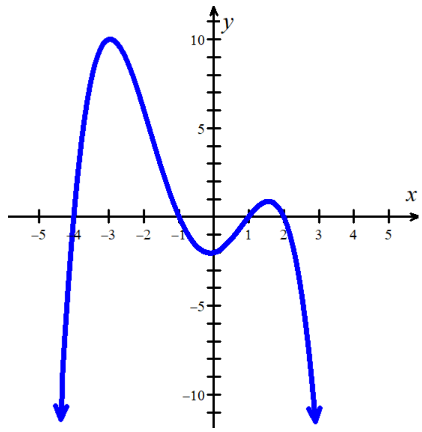
28. Write an equation for the polynomial $p(x)$, of degree 5, that is graphed below. Leave your equation in factored form.

Figure Ex2.2. 1



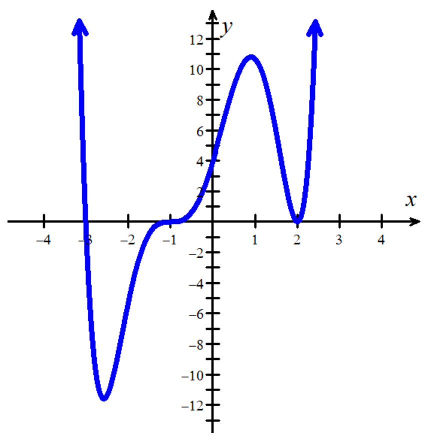
29. Write an equation for the polynomial $p(x)$, of degree 4, that is graphed below. Leave your equation in factored form.

Figure Ex2.2. 2



30. Write an equation for the polynomial $p(x)$, of degree 6, that is graphed below. Note that the zero $x = -1$ has multiplicity 3. Leave your equation in factored form.

Figure Ex2.2. 3



In Exercises 31 – 36, given the pair of functions f and g, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and applying transformations. Track at least three points of your choice through the transformations. State the domain and range of g.

31. $f(x) = x^3,\ g(x) = (x+2)^3 + 1$

32. $f(x) = x^4,\ g(x) = (x+2)^4 + 1$

33. $f(x) = x^4,\ g(x) = 2 - 3(x-1)^4$

34. $f(x) = x^5,\ g(x) = -x^5 - 3$

35. $f(x) = x^5,\ g(x) = (x+1)^5 + 10$

36. $f(x) = x^6,\ g(x) = 8 - x^6$

37. Use the Intermediate Value Theorem to prove that $f(x)=x^3-9x+5$ has a real zero in each of the following intervals: $(-4,-3)$, $(0,1)$ and $(2,3)$.

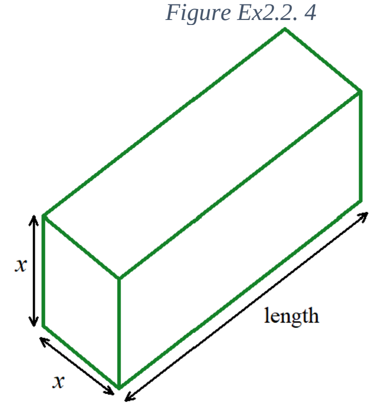
38. Use the Intermediate Value Theorem to confirm that $f(x)=x^3-100x+2$ has at least one real zero between $x=0.01$ and $x=0.1$.

39. Show that the function $f(x)=x^3-5x^2+3x+6$ has at least two real zeros between $x=1$ and $x=4$.

40. Rework **Example 2.2.6**, assuming the box is to be made from an 8.5 inch by 11 inch sheet of paper. Then, using scissors and tape, construct the box. Are you surprised?

41. According to US Postal regulations, a rectangular shipping box must satisfy the inequality $\text{Length}+\text{Girth}\leq 130 \text{ inches}$ for Parcel Post and $\text{Length}+\text{Girth}\leq 108 \text{ inches}$ when using a private company's shipping services. Let's assume we have a closed rectangular box with a square face of side length x, as drawn below. The length is the longest side and is clearly labeled. The girth is the distance around the box in the other two dimensions so, in our case, it is the sum of the four sides of the square, $4x$.

Figure Ex2.2. 4



(a) Assuming that we'll be mailing a box via Parcel Post where $\text{Length}+\text{Girth}=130 \text{ inches}$, express the length of the box in terms of x and then express the volume V of the box in terms of x.

(b) Find the dimensions of the box of maximum volume that can be shipped via Parcel Post.

(c) Repeat parts (a) and (b) if the box is shipped using a private company's shipping services.

42. Suppose the profit P, in thousands of dollars, from producing and selling x hundred LCD TV's is given by $P(x)=-5x^3+35x^2-45x-25$ for $0\leq x\leq 10.07$. Use graphing technology to graph $y=P(x)$ and determine the number of TV's that should be sold to maximize profit. What is the maximum profit?

43. While developing their newest game, Sasquatch Attack!, the makers of the PortaBoy revised their profit function and now use $P(x) = -0.03x^3 + 3x^2 + 25x - 250$, for $x \geq 0$. Use graphing technology to find the production level x that maximizes the profit made by producing and selling x PortaBoy game systems.

44. Show that the end behavior of a linear function $f(x) = mx + b$, $m \neq 0$, is as it should be according to the results we've established in this section for polynomials of odd degree. (That is, show that the graph of a linear function is 'up on one side' and 'down on the other' just like the graph of $y = a_n x^n$ for odd numbers n.)

45. Here are a few questions for you to discuss with your classmates.

(a) How many local extrema could a polynomial of degree n have? How few local extrema?

(b) Could a polynomial have two local maxima but no local minima?

(c) If a polynomial has two local maxima and two local minima, can it be of odd degree? Can it be of even degree?

(d) Can a polynomial have local extrema without having any real zeros?

(e) Why must every polynomial of odd degree have at least one real zero?

(f) Can a polynomial have two distinct real zeros and no local extrema?

(g) Can an *x*-intercept yield a local extremum? Can it yield an absolute extremum?

(h) If the *y*-intercept yields an absolute minimum, what can we say about the degree of the polynomial and the sign of the leading coefficient?

2.3 Using Synthetic Division to Factor Polynomials

Learning Objectives

- Use division to factor polynomials and determine zeros.
- Use synthetic division to simplify the division process.
- Use the Remainder Theorem to find function values of polynomials.
- Use the Factor Theorem to relate zeros to factors of polynomials.

Using Division to Find Zeros of Polynomials

Suppose we wish to find the zeros of $f(x)=x^3+4x^2-5x-14$. Setting $f(x)=0$ results in the polynomial equation $x^3+4x^2-5x-14=0$. Despite all of the factoring techniques we learned in Intermediate Algebra, this equation foils[13] us at every turn. Should we happen to guess (correctly) that $x=2$ is a zero, there must be a factor of $(x-2)$ lurking around in the factorization of $f(x)$. How could we use the factor $(x-2)$ to find this factorization? The answer comes from our old friend, polynomial division. Dividing $x^3+4x^2-5x-14$ by $x-2$ gives

$$\begin{array}{r}
x^2+6x+7 \\
x-2\overline{\big)\,x^3+4x^2-5x-14} \\
-\underline{\left(x^3-2x^2\right)} \qquad\quad \\
6x^2-5x \qquad\ \\
-\underline{\left(6x^2-12x\right)} \qquad \\
7x-14 \\
-\underline{\left(7x-14\right)} \\
0
\end{array}$$

This means $\left(x^3+4x^2-5x-14\right)\div(x-2)=x^2+6x+7$ or, after multiplying both sides by $(x-2)$, $x^3+4x^2-5x-14=\left(x^2+6x+7\right)(x-2)$.

[13] Pun intended.

2.3 Exercises

1. If division of a polynomial (of degree at least 1) by $x+4$ results in a remainder of zero, what can we conclude?
2. If a polynomial of degree n is divided by a binomial of degree 1, what is the degree of the quotient?

In Exercises 3 – 8, use polynomial long division to perform the indicated division. Write the polynomial in the form $p(x)=d(x)q(x)+r(x)$.

3. $\left(4x^2+3x-1\right)\div(x-3)$
4. $\left(2x^3-x+1\right)\div\left(x^2+x+1\right)$
5. $\left(5x^4-3x^3+2x^2-1\right)\div\left(x^2+4\right)$
6. $\left(-x^5+7x^3-x\right)\div\left(x^3-x^2+1\right)$
7. $\left(9x^3+5\right)\div(2x-3)$
8. $\left(4x^2-x-23\right)\div\left(x^2-1\right)$

In Exercises 9 – 22, use synthetic division to perform the indicated division. Write the polynomial in the form $p(x)=d(x)q(x)+r(x)$.

9. $\left(3x^2-2x+1\right)\div(x-1)$
10. $\left(x^2-5\right)\div(x-5)$
11. $\left(3-4x-2x^2\right)\div(x+1)$
12. $\left(4x^2-5x+3\right)\div(x+3)$
13. $\left(x^3+8\right)\div(x+2)$
14. $\left(4x^3+2x-3\right)\div(x-3)$
15. $\left(18x^2-15x-25\right)\div\left(x-\frac{5}{3}\right)$
16. $\left(4x^2-1\right)\div\left(x-\frac{1}{2}\right)$
17. $\left(2x^3+x^2+2x+1\right)\div\left(x+\frac{1}{2}\right)$
18. $\left(3x^3-x+4\right)\div\left(x-\frac{2}{3}\right)$
19. $\left(2x^3-3x+1\right)\div\left(x-\frac{1}{2}\right)$
20. $\left(4x^4-12x^3+13x^2-12x+9\right)\div\left(x-\frac{3}{2}\right)$
21. $\left(x^4-6x^2+9\right)\div\left(x-\sqrt{3}\right)$
22. $\left(x^6-6x^4+12x^2-8\right)\div\left(x+\sqrt{2}\right)$

In Exercises 23 – 32, determine $p(c)$ using the Remainder Theorem for the given polynomial function and value of c. If $p(c)=0$, write the polynomial in the factored form $p(x)=(x-c)q(x)$.

23. $p(x)=2x^2-x+1$, $c=4$
24. $p(x)=4x^2-33x-180$, $c=12$

25. $p(x)=2x^3-x+6$, $c=-3$

26. $p(x)=x^3+2x^2+3x+4$, $c=-1$

27. $p(x)=3x^3-6x^2+4x-8$, $c=2$

28. $p(x)=8x^3+12x^2+6x+1$, $c=-\frac{1}{2}$

29. $p(x)=x^4-2x^2+4$, $c=\frac{3}{2}$

30. $p(x)=6x^4-x^2+2$, $c=-\frac{2}{3}$

31. $p(x)=x^4+x^3-6x^2-7x-7$, $c=-\sqrt{7}$

32. $p(x)=x^2-4x+1$, $c=2-\sqrt{3}$

In Exercises 33 – 44, you are given a polynomial equation along with one or more of its zeros, one of its factors, or its graph. Use the techniques of this section to find the remaining real zeros and factor the polynomial.

33. $p(x)=x^3-6x^2+11x-6$, zero $x=1$

34. $p(x)=x^3-24x^2+192x-512$, factor $x-8$

35. $p(x)=2x^3-3x^2-11x+6$, zero $x=\frac{1}{2}$

36. $p(x)=2x^3-x^2-10x+5$, factor $x-\frac{1}{2}$

37. $p(x)=3x^4+10x^3+7x^2-4x-4$, zeros $x=\frac{2}{3}$ and $x=-2$

38. $p(x)=x^4-2x^3-11x^2+6x+24$, zeros $x=-2$ and $x=4$

39. $p(x)=4x^4-28x^3+61x^2-42x+9$, zero $x=\frac{1}{2}$ of multiplicity 2

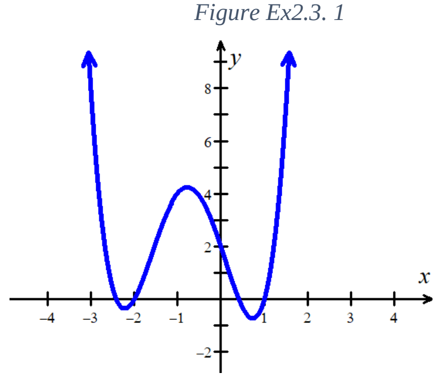
40. $p(x)=x^5+2x^4-12x^3-38x^2-37x-12$, zero $x=-1$ of multiplicity 3

41. $p(x)=125x^5-275x^4-2265x^3-3213x^2-1728x-324$, zero $x=-\frac{3}{5}$ of multiplicity 3

42. $p(x)=x^2-2x-2$, zero $x=1-\sqrt{3}$

43. $p(x)=x^4+3x^3-x^2-5x+2$

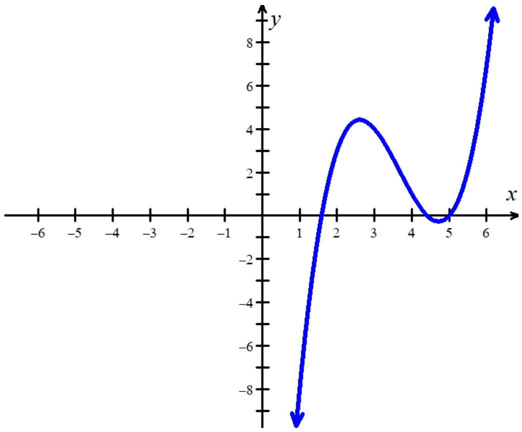
Figure Ex2.3. 1



Graph of $p(x)=x^4+3x^3-x^2-5x+2$

44. $p(x)=x^3-11x^2+37x-35$

Figure Ex2.3. 2



Graph of $p(x)=x^3-11x^2+37x-35$

45. Find a quadratic polynomial with integer coefficients that has $x=\frac{3}{5}\pm\frac{\sqrt{29}}{5}$ as its real zeros.

2.4 Exercises

1. Explain why the Rational Zeros Theorem does not guarantee finding zeros of a polynomial function.
2. If synthetic division reveals a zero, why should we try that value again as a possible solution?

In Exercises 3 – 12, for the given polynomial, use the Rational Zeros Theorem to make a list of possible rational zeros.

3. $f(x)=x^3-2x^2-5x+6$
4. $f(x)=x^4+2x^3-12x^2-40x-32$
5. $f(x)=x^4-9x^2-4x+12$
6. $f(x)=x^3+4x^2-11x+6$
7. $f(x)=x^3-7x^2+x-7$
8. $f(x)=-2x^3+19x^2-49x+20$
9. $f(x)=-17x^3+5x^2+34x-10$
10. $f(x)=36x^4-12x^3-11x^2+2x+1$
11. $f(x)=3x^3+3x^2-11x-10$
12. $f(x)=2x^4+x^3-7x^2-3x+3$

In Exercises 13 – 32, find the real zeros of the polynomial. State the multiplicity of each real zero.

13. $f(x)=x^3-2x^2-5x+6$
14. $f(x)=x^4+2x^3-12x^2-40x-32$
15. $f(x)=x^4-9x^2-4x+12$
16. $f(x)=x^3+4x^2-11x+6$
17. $f(x)=x^3-7x^2+x-7$
18. $f(x)=-2x^3+19x^2-49x+20$
19. $f(x)=-17x^3+5x^2+34x-10$
20. $f(x)=36x^4-12x^3-11x^2+2x+1$
21. $f(x)=3x^3+3x^2-11x-10$
22. $f(x)=2x^4+x^3-7x^2-3x+3$
23. $f(x)=9x^3-5x^2-x$
24. $f(x)=6x^4-5x^3-9x^2$
25. $f(x)=x^4+2x^2-15$
26. $f(x)=x^4-9x^2+14$
27. $f(x)=3x^4-14x^2-5$
28. $f(x)=2x^4-7x^2+6$
29. $f(x)=x^6-3x^3-10$
30. $f(x)=2x^6-9x^3+10$
31. $f(x)=x^5-2x^4-4x+8$
32. $f(x)=2x^5+3x^4-18x-27$

In Exercises 33 – 42, find the real solutions of the polynomial equation.

33. $9x^3 = 5x^2 + x$

34. $9x^2 + 5x^3 = 6x^4$

35. $x^3 + 6 = 2x^2 + 5x$

36. $x^4 + 2x^3 = 12x^2 + 40x + 32$

37. $x^3 - 7x^2 = 7 - x$

38. $2x^3 = 19x^2 - 49x + 20$

39. $x^3 + x^2 = \dfrac{11x + 10}{3}$

40. $x^4 + 2x^2 = 15$

41. $14x^2 + 5 = 3x^4$

42. $2x^5 + 3x^4 = 18x + 27$

In Exercises 43 – 52, use Descartes' Rule of Signs (shown below) to list the possible number of positive and negative real zeros. Compare your results with solutions to Exercises 13 through 22.

Descartes' Rule of Signs: Suppose $f(x)$ is the formula for a polynomial function written with descending powers of x.

- If P denotes the number of variations of sign in the formula for $f(x)$, then the number of positive real zeros (counting multiplicities) is one of the numbers $\{P, P-2, P-4, \ldots\}$.
- If N denotes the number of variations of sign in the formula for $f(-x)$, then the number of negative real zeros (counting multiplicities) is one of the numbers $\{N, N-2, N-4, \ldots\}$.

Notes:

1. To determine variations in sign, consider $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. If we focus only on the **signs** of the coefficients, we start with a (+), followed by another (+), then switch to (–), and stay (–) for the remaining two coefficients. Since the signs of the coefficients switched **once** as we read from left to right, we say that $f(x)$ has **one** variation in sign.
2. The number of positive or negative real zeros always starts with the number of sign changes and decreases by an even number. For example, if $f(x)$ has 7 sign changes, then, counting multiplicities, f has 7, 5, 3, or 1 positive real zeros. If $f(-x)$ results in 4 sign changes, then, counting multiplicities, f has 4, 2, or 0 negative real zeros.

43. $f(x) = x^3 - 2x^2 - 5x + 6$

44. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$

45. $f(x) = x^4 - 9x^2 - 4x + 12$

46. $f(x) = x^3 + 4x^2 - 11x + 6$

47. $f(x) = x^3 - 7x^2 + x - 7$

48. $f(x) = -2x^3 + 19x^2 - 49x + 20$

49. $f(x) = -17x^3 + 5x^2 + 34x - 10$

50. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$

51. $f(x) = 3x^3 + 3x^2 - 11x - 10$

52. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$

53. Let $f(x) = 5x^7 - 33x^6 + 3x^5 - 71x^4 - 597x^3 + 2097x^2 - 1971x + 567$. With the help of your classmates, find the *x*- and *y*-intercepts of the graph of f. Find the intervals on which the graph is above the *x*-axis and the intervals on which the graph is below the *x*-axis. Sketch a rough graph of f.

2.5 Exercises

1. If $p(x)$ is a polynomial with real number coefficients and $-2i$ is a zero of $p(x)$, what do we know about the factors of $p(x)$?

2. If $p(x)$ is a polynomial with real number coefficients and no real zeros, can any conclusions be drawn about the end behavior of p?

In Exercises 3 – 12, use the given complex numbers z and w to find and simplify the following. Write your answers in the form $a+bi$.

(a) $z+w$ (b) $z\cdot w$ (c) z^2

(d) $\dfrac{z}{w}$ (e) the conjugate of z (f) $z\cdot(\text{the conjugate of } z)$

3. $z=2+3i$, $w=4i$
4. $z=1+i$, $w=-i$
5. $z=i$, $w=-1+2i$
6. $z=4i$, $w=2-2i$
7. $z=3-5i$, $w=2+7i$
8. $z=-5+i$, $w=4+2i$
9. $z=\sqrt{2}-\sqrt{2}i$, $w=\sqrt{2}+\sqrt{2}i$
10. $z=1-\sqrt{3}i$, $w=-1-\sqrt{3}i$
11. $z=\frac{1}{2}+\frac{\sqrt{3}}{2}i$, $w=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$
12. $z=-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$, $w=-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$

In Exercises 13 – 20, simplify the quantity.

13. $\sqrt{-49}$
14. $\sqrt{-9}$
15. $\sqrt{-25}\sqrt{-4}$
16. $\sqrt{(-25)(-4)}$
17. $\sqrt{-9}\sqrt{-16}$
18. $\sqrt{(-9)(-16)}$
19. $\sqrt{-(-9)}$
20. $-\sqrt{-9}$

We know that $i^2=-1$ which means $i^3=i^2\cdot i=(-1)i=-i$ and $i^4=i^2\cdot i^2=(-1)(-1)=1$. In Exercises 21 – 28, use this information to simplify the given power of i.

21. i^5
22. i^6
23. i^{-7}
24. i^8
25. i^{15}
26. i^{26}
27. i^{117}
28. i^{304}

In Exercises 29 – 34, find all zeros of the polynomial and then completely factor it to linear and irreducible quadratic factors.

29. $f(x)=x^3+3x^2+4x+12$

30. $f(x)=x^3-2x^2+9x-18$

31. $f(x)=3x^3-13x^2+43x-13$

32. $f(x)=x^3+6x^2+6x+5$

33. $f(x)=4x^4-4x^3+13x^2-12x+3$

34. $f(x)=2x^4-7x^3+14x^2-15x+6$

In Exercises 35 – 50, find all zeros of the polynomial.

35. $f(x)=x^2-4x+13$

36. $f(x)=3x^2+2x+10$

37. $f(x)=x^2-2x+5$

38. $f(x)=9x^3+2x+1$

39. $f(x)=4x^3-6x^2-8x+15$ (Hint: $x=-\frac{3}{2}$ is one of the zeros.)

40. $f(x)=6x^4+17x^3-55x^2+16x+12$ (Hint: $x=\frac{3}{2}$ is one of the zeros.)

41. $f(x)=8x^4+50x^3+43x^2+2x-4$ (Hint: $x=-\frac{1}{2}$ is one of the zeros.)

42. $f(x)=x^3+7x^2+9x-2$

43. $f(x)=x^4+x^3+7x^2+9x-18$

44. $f(x)=-3x^4-8x^3-12x^2-12x-5$

45. $f(x)=x^4+9x^2+20$

46. $f(x)=x^4+5x^2-24$

47. $f(x)=x^6-64$

48. $f(x)=x^5-x^4+7x^3-7x^2+12x-12$

49. $f(x)=x^4-2x^3+27x^2-2x+26$ (Hint: $x=i$ is one of the zeros.)

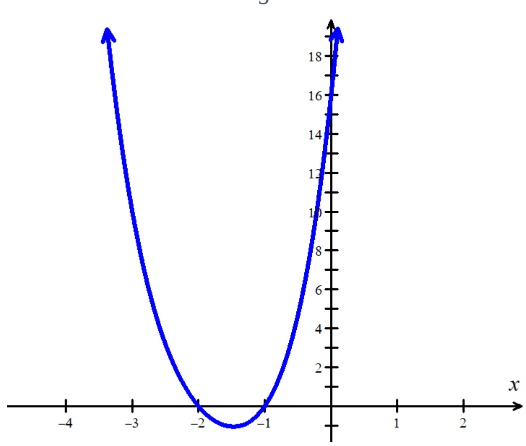
50. $f(x)=2x^4+5x^3+13x^2+7x+5$ (Hint: $x=-1+2i$ is a zero.)

In Exercises 51 – 55, create a polynomial f with real number coefficients that has all desired characteristics. You may leave the polynomial in factored form.

51. • The zeros of f are $x=\pm 1$ and $x=\pm i$.
   • The leading term of $f(x)$ is $42x^4$.
52. • $x=2i$ is a zero.
   • The point $(-1,0)$ is a local minimum on the graph of $y=f(x)$.
   • The leading term of $f(x)$ is $117x^4$.
53. • The solutions to $f(x)=0$ are $x=\pm 2$ and $x=\pm 7i$.
   • The leading term of $f(x)$ is $-3x^5$.
   • The point $(2,0)$ is a local maximum on the graph of $y=f(x)$.
54. • f is degree 5.
   • $x=6$, $x=i$ and $x=1-3i$ are zeros of f.
   • As $x\to-\infty$, $f(x)\to\infty$.
55. • The leading term of $f(x)$ is $-2x^3$.
   • $x=2i$ is a zero.
   • $f(0)=-16$.

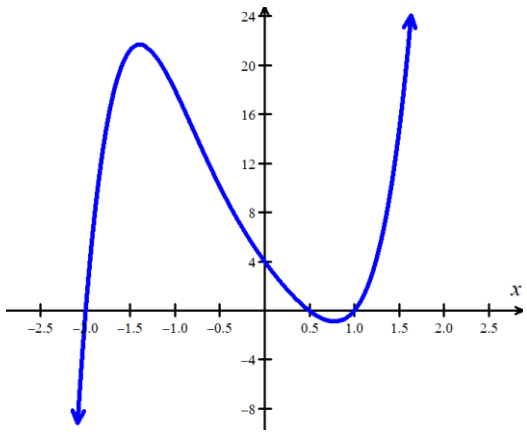
56. Find all zeros and completely factor $f(x)=x^4+7x^3+22x^2+32x+16$ to linear and irreducible quadratic factors. Give exact values, using radicals where necessary. The graph of $f(x)$ is below.

Figure Ex2.5. 1



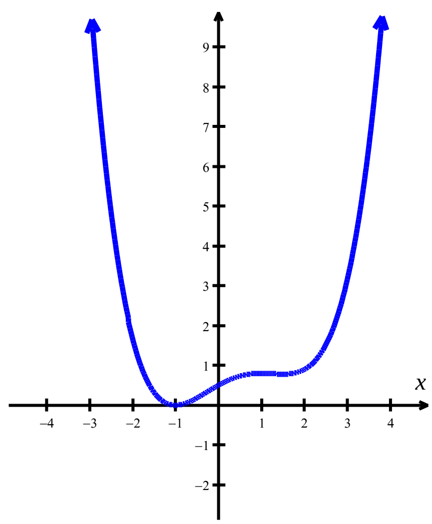
57. Find all zeros and completely factor $f(x) = 2x^5 + x^4 - x^3 + 4x^2 - 10x + 4$ to linear and irreducible quadratic factors. Give exact values, using radicals where necessary. The graph of $f(x)$ is shown below.

Figure Ex2.5. 2



58. Find a polynomial function $P(x)$ of degree 4 with real coefficients that has a zero of $x = 2 + i$ and a leading coefficient of $\frac{1}{10}$. A graph of $P(x)$ is shown below. Write the function $P(x)$ in factored form.

Figure Ex2.5. 3



2.6 Polynomial Inequalities

Learning Objectives

- Solve polynomial inequalities graphically.
- Solve polynomial inequalities analytically.

In this section, we develop techniques for solving polynomial inequalities. We determine solutions analytically, and look at them graphically. This first example motivates the core ideas.

Example 2.6.1. Let $f(x)=2x-1$ and $g(x)=5$.

1. Solve $f(x)=g(x)$. 2. Solve $f(x)<g(x)$. 3. Solve $f(x)>g(x)$.

4. Draw the graphs of $y=f(x)$ and $y=g(x)$ on the same set of axes and interpret your solutions to parts 1 through 3 above.

Solution.

1. Replacing f and g by their definitions in the equation $f(x)=g(x)$, we have $2x-1=5$. Solving this equation, we get $x=3$.

2. The inequality $f(x)<g(x)$ is equivalent to $2x-1<5$. The solution of this inequality is $x<3$, or all numbers less than 3. In set-builder notation, the answer is $\{x \mid x<3\}$. The answer in interval notation is $(-\infty,3)$. The answer may be stated in either format, unless a specific format is required.

3. To find where $f(x)>g(x)$, we solve $2x-1>5$. We get $x>3$, or the interval $(3,\infty)$.

4. To draw the graph of $y=f(x)$, we plot $y=2x-1$, which is a line with a y-intercept of $(0,-1)$ and a slope of 2. The graph of $y=g(x)$ is the graph of $y=5$, which is a horizontal line through $(0,5)$.

2.6 Exercises

1. In creating a sign chart, how many test values must be used to determine the solution of an inequality whose highest power term is of degree 3?
2. If a polynomial inequality has no solutions, what can be said about the graphs of the two sides of the inequality?

In Exercises 3 – 26, solve the inequality. Write your answer using interval notation.

3. $(x-7)(x+2)<0$
4. $-3(x+4)(x+5)\geq 0$
5. $x^2+2x-3\geq 0$
6. $16x^2+8x+1>0$
7. $x^2+9<6x$
8. $9x^2+16\geq 24x$
9. $x^2+4\leq 4x$
10. $x^2+1<0$
11. $3x^2\leq 11x+4$
12. $x>x^2$
13. $2x^2-4x-1>0$
14. $5x+4\leq 3x^2$
15. $x^2+x+1\geq 0$
16. $(x-9)(x+11)(x-7)^2<0$
17. $-2(x-3)(x-1)(x+7)\leq 0$
18. $x^4-9x^2\leq 4x-12$
19. $(x-1)^2\geq 4$
20. $4x^3\geq 3x+1$
21. $x^4\leq 16+4x-x^3$
22. $3x^2+2x<x^4$
23. $\dfrac{x^3+2x^2}{2}<x+2$
24. $\dfrac{x^3+20x}{8}\geq x^2+2$
25. $2x^4>5x^2+3$
26. $x^6+x^3\geq 6$

27. The profit, in dollars, made by selling x bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is given by $P(x)=-x^2+25x-100$, for $0\leq x\leq 35$. How many bottles of tonic must be sold to make at least \$50 in profit?

28. Suppose $C(x)=x^2-10x+27$, $3\leq x\leq 15$, represents the marginal cost in hundreds of dollars to produce an additional x thousand pens. Find how many additional pens can be produced with marginal cost of no more than \$1100. Give your answer as an interval of values.

29. Suppose $d(x) = 0.04x^2 + 0.6x$ represents the stopping distance in feet of a car from the speed of x miles per hour. Find the maximum speed from which the car stops in no more than 130 feet.

30. The temperature T, in degrees Fahrenheit, t hours after 6 AM, is given by $T(t) = -\frac{1}{2}t^2 + 8t + 32$ for $0 \le t \le 12$. When is it warmer than 42° Fahrenheit?

31. The height h in feet of a model rocket above the ground t seconds after lift-off is given by $h(t) = -5t^2 + 100t$ for $0 \le t \le 20$. When is the rocket at least 250 feet off the ground? Round your answer to two decimal places.

32. Let $f(x) = 3x^{10} - 7x^9 - 6x^8 + 30x^7 - 19x^6 - 29x^5 + 34x^4 + 6x^3 - 12x^2$. Use the fact that $f(1-i) = 0$ to solve the inequality $f(x) \ge 0$. (Hint: Start with finding rational zeros.)